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Integral Equations For Inhomogeneous Magnetoplasma Waves

by

Paul Diament

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INTEGRAL EQUATIONS FOR
INHOMOGENEOUS MAGNETOPLASMA WAVES

Paul Diament*

Institute for Plasma Research
Stanford University
Stanford, California

ABSTRACT

The problem of wave dispersion and stability for a class of hot, inhomogeneous, collisionless magnetoplasmas is reduced to the solution of an integral equation with well-behaved kernel. Admissible configurations include those for which the externally applied and internal ambipolar fields form a generalized harmonic oscillator. The full set of Maxwell's equations is used to arrive at self-consistent perturbation fields in terms of the equilibrium particle distributions. An illustrative example treats a magnetoplasma column with Gaussian radial profile and Maxwellian velocity distribution in a state of quasi-equilibrium.

*On leave from the Department of Electrical Engineering, Columbia University, New York, 1966-67.

CONTENTS

I. INTRODUCTION

The propagation and evolution of small disturbances in a hot, magnetized, inhomogeneous plasma is a matter of considerable theoretical and practical interest with regard to both the wave supporting capabilities of the medium and its stability against small perturbations. Although the procedure leading to descriptions and predictions of the response of the plasma to small signals can be formulated quite explicitly, the problem has long resisted definitive analysis, by virtue of what appears to be an intrinsic incompatibility with the ideal conditions under which standard analyses and perturbation techniques are effective. The major complicating features are the nonlinearity of the system dynamics and the inhomogeneity of the medium, which makes the system sensitive, via particle transport, to global as well as local conditions.

The general analysis procedure can be decomposed into a number of distinct subproblems, each of which isolates a particular feature of the system, such as inhomogeneity, nonlinearity, anisotropy, statistical distribution, electromagnetic interaction. These may be dealt with individually at first, then combined in a self-consistent manner. The concatenation of the subproblems links the tractability of one quite strongly to that of the preceding one, with the result that the complexity of the mathematical description forces recourse to numerical analysis at a very early stage of the calculation. It soon becomes hopeless to direct and interpret a realistic computation meaningfully. It is the purpose of this paper to present an approach to the problem which can be pushed analytically nearly to completion, leaving to digital computers only some relatively straightforward quadratures.

The system dealt with is, in the general formulation, a collection of charged particles of two species, inhomogeneously distributed in space, with arbitrary velocity distributions, immersed in a uniform and constant applied magnetic field, in the absence of collisions. The evolution of arbitrary weak disturbances of an equilibrium state is sought, for the determination of stability and wave dispersion. The problem is three-dimensional and is treated nonrelativistically, but with the full set of Maxwell's equations and retaining the r.f. magnetic field interaction. A particular class of plasma configurations is considered, whose realizability is discussed and justified on physical grounds.

Experiments on inhomogeneous magnetoplasmas were reported and interpreted by Buchsbaum and Hasegawa.[1,2] They obtained a fourth-order differential equation for the potential, without accounting for the ambipolar fields. Pearson[3,4] extended their treatment by allowing for anisotropic velocity distributions and ambipolar fields. Baldwin[5] considers wave propagation in inhomogeneous magnetoplasmas in the low temperature limit, with no resonant particle effects. Analyses such as these inherently lack generality, being based on various approximations which effectively remove the nonlocal nature of the interactions, or require low temperature or only slight inhomogeneity, or rely on quasistatic conditions. They can be quite successful in interpreting experiments that conform to the specialized assumed conditions, although the results may often be valid in only highly restricted ranges of the parameters, such as near the second cyclotron harmonic. Due recognition of nonlocal interactions was given by Buneman[6] in his analysis of the Bennett pinch. This leads to an integral equation, rather than differential equations, but one of a rather intractable character, due to the complexity of the particle orbits and to the effects of particle motions in resonance with the disturbances. The present work deals with a class of systems describable by tractable integral equations.

II. OUTLINE OF METHOD

Starting with a specified particle distribution function $f_o(\underset{\sim}{x},\underset{\sim}{v})$, for each species, at time $t = 0$, the problem may be stated as that of determining its subsequent time development, $f(\underset{\sim}{x},\underset{\sim}{v},t)$, since this function provides statistical information as to the state of the system. The particles so described are subjected to forces due to the applied magnetic field, the space charge forces of the ambipolar electric field, the self magnetic field due to unbalanced currents, and to small-signal electromagnetic fields accompanying a weak disturbance, either applied externally or self-generated. Collisions are excluded from consideration in this paper.

The evolution of the initial distribution function is implicit in the Boltzmann equation, which states that the density in phase space remains constant along the particle trajectories, in the absence of collisions. Regardless of how the equation is formulated, its solution entails the determination of the particle orbits, under the action of all the forces. The distribution at time t is determined by evaluating the initial one at that point in phase space, $(\underset{\sim}{x}_o,\underset{\sim}{v}_o)$, where a particle arriving at $\underset{\sim}{x}$ with velocity $\underset{\sim}{v}$ at time $\underset{\sim}{t}$ originated at $t = 0$. Since the forces other than the applied ones are themselves determined from sources obtained by suitable averaging over the distribution, the overall problem is nonlinear. This stage of the analysis is hence to be carried out implicitly to obtain expressions for the particle orbits, particularly the initial phase points, as functionals of unknown force fields, to be found subsequently from the functional form of the distribution. Clearly, it is essential to arrive at a fairly explicit form of the particle trajectories in order to permit the requisite processing of the distribution function to extract the unknown fie'ds consistently.

Two intermediate steps are involved before the closing of the set of equations through the extraction of self-consistent fields. First, the distribution $f(\underset{\sim}{x},\underset{\sim}{v},t)$, as yet found only implicitly in terms of unknown force fields, is to be suitably weighted and averaged to obtain the electric charge and current distributions, still in terms of the electromagnetic fields of which they are the sources. This step represents the determination of the constitutive parameters of the medium. The next step is the solution of Maxwell's equations, with these charges and currents as sources, to obtain the fields which exert the originally assumed forces. The condition of self-consistency then expresses the dispersion relation for the medium, including the stability of the unperturbed state.

The anisotropy of the magnetoplasma results in the tensor character of the constitutive parameters, so that the entire analysis is best dealt with by matrix calculus. This minor complication is overshadowed by the further one that, in the inhomogeneous medium, the constitutive parameters are not expressible simply as permittivity or conductivity tensors but, more generally, as integral operators expressing the non-local nature of the medium's sensitivity to disturbances. This is a direct result of the transport of particles along orbits that span regions of varying properties, with different responses to perturbations. This global sensitivity ultimately leads to an integral equation for the self-consistent fields.

It is clear then that for this program not to be frustrated from the start, it is essential that the particle transport be expressible analytically, and in relatively simple form, rather than merely numerically. This is virtually impossible to achieve in a completely self-consistent manner, however, for the particle trajectories are generally highly complicated solutions to nonlinear differential equations. In particular, the orbits in the inhomogeneous magnetoplasma are not simply helical, because of the action of the ambipolar electric field. Thus, the description of even the equilibrium state is a difficult task in fully realistic situations.

The key to the analysis to follow is the construction of a plasma configuration which is characterized by simply expressed unperturbed particle orbits. The class of orbits to be admitted includes those which solve generalized harmonic oscillator equations. Although the associated configurations are found in the general inhomogeneous, two-species case not to be strictly self-consistent and time-invariant, it will be shown that, by proper choice of the parameters, the initial state can be made to persist for relatively long periods of time, if not indefinitely. The effects which ultimately unravel the quasi-equilibrium unperturbed state can be made second order and act on a relatively long time scale. Since the restriction to collision-free systems has already limited the time duration of validity of the analysis, there is no inconsistency in investigating the short-term stability of a slowly varying unperturbed state.

Even with simple analytic expressions for the particle orbits, the standard approach based on the distribution function would lead ultimately to an integral equation whose kernel exhibits certain singularities. These are associated with particles whose orbits are in resonance with the propagating disturbances and are essential to the stability problem in that they lead to collisionless damping or growth phenomena. The standard procedure for dealing with the singularities, as formulated by Landau,[7] involves complex contour integrations that are inimical to convenient numerical analysis.

To avoid the appearance of singular kernels, the method to be used describes the system in terms of its "inverse phase space spectrum," i.e. through the Fourier transform of the particle distribution function. This has a number of additional advantages, in that it is an algebraically simpler description which virtually eliminates the two intermediate steps of the program outlined above. The extraction of the sources of the perturbation fields involves simply evaluation or differentiation of the spectrum, rather than integration of the distribution over velocity. In addition, the transform formulation makes the solution of Maxwell's equations merely a matter of algebraic manipulation. As a result, the procedure can be carried forth analytically to a final integral equation with nonsingular kernel, which may be solved numerically in a straightforward manner.

As an illustration of the method, the integral equation expressing the dispersion relation of the medium is obtained for a plasma with a Maxwellian distribution in velocity and a Gaussian distribution in space.

III. EVOLUTION OF INITIAL STATE

For compactness of notation in the initial manipulations, phase space will be considered as that spanned by the six-vector ϕ, comprised of the two position vectors in configuration and velocity space, $\underset{\sim}{x}, \underset{\sim}{v}$. The first task is the determination of the particle distribution function $f(\phi,t)$, given the initial one, $f(\phi,0) = f_o(\phi)$.

In the absence of collisions, the Boltzmann equation prescribes that

$$f(\phi,t)\, d^6\phi = f_o(\phi_o)\, d^6\phi_o \quad , \tag{1}$$

where ϕ_o is the initial phase point of a particle whose phase point is ϕ at time t. This expresses particle conservation, or continuity in phase space along the particle trajectories.

Consider any system for which the zero-order orbit equation is linear in the phase; i.e. has the form of a generalized harmonic oscillator equation:

$$d\phi/dt = Y\, \phi(t) \quad . \tag{2}$$

It is assumed here that the 6 × 6 matrix Y is constant. determined by both the externally applied force field and the internal space charge fields associated with the inhomogeneous distribution. It will be shown later that this assumption is exact for a homogeneous magnetoplasma and very nearly exact for an inhomogeneous one.

Under the action of the perturbation, the orbit equation includes an anharmonic term, due to the additional acceleration built into the six-vector $a_6(\phi,t)$. The trajectory is then specified by the dynamic equation,

$$d\phi(\lambda)/d\lambda = Y\ \phi(\lambda) + a_6(\phi(\lambda),\lambda), \qquad \phi(t) = \phi\ , \tag{3}$$

in which the "initial" condition directs the orbit to the phase point ϕ at time t . What is required in (1) is the initial phase $\phi_o = \phi(0)$.

An equivalent integral equation for the orbit is

$$\phi(\lambda) = e^{-Y(t-\lambda)}\phi - \int_\lambda^t e^{-Y(\tau-\lambda)} a_6(\phi(\tau),\tau)\ d\tau\ , \tag{4}$$

and the initial phase is given in terms of its solution $\phi(\lambda)$ by

$$\phi_o = e^{-Yt}\ \phi - \int_o^t e^{-Y\tau} a_6(\phi(\tau),\tau)\ d\tau\ . \tag{5}$$

Specializing these exact, general equations to the case of weak perturbations of the harmonic orbits, the initial phase $\phi_o = \phi_o(\phi,t)$ is given to first order in the perturbation $a_6(\phi,t)$ by

$$\phi_o = e^{-Yt}\left[\phi - \int_o^t e^{Y\lambda}\ a_6(e^{-Y\lambda}\phi, t-\lambda)\ d\lambda\right]. \tag{6}$$

The assumption of a harmonic system has led to a simple, explicit expression for the initial phase point of the perturbed orbits for any weak disturbance.

To complete the determination of the time development of the initial distribution, there remains to relate the initial and final elements of phase space, $d^6\phi_o$ and $d^6\phi$. The relation is given by the Jacobian of the transformation:

$$d^6\phi_o = |\det \partial\phi_o/\partial\phi|\ d^6\phi \quad . \tag{7}$$

But the acceleration affecting the particles is quite generally solenoidal in velocity, with the consequence that the force fields are such that $\mathrm{Tr}\ Y = 0$ and $(\partial/\partial\phi)\cdot a_6(\phi,t) = 0$, at least in the absence of collisions. As a result, (6) leads, at least to first order, to $\det\ (\partial\phi_o/\partial\phi) = 1$, so that $d^6\phi_o = d^6\phi$. The element of phase space is invariant.

The time development of the initial distribution is thus expressed to first order in the perturbation by

$$f(\phi,t) = f_o\left(e^{-Yt}\left[\phi - \int_o^t e^{Y\lambda} a_6(e^{-Y\lambda}\phi, t-\lambda)\ d\lambda\right]\right) \quad . \tag{8}$$

Assuming further that the initial distribution is an equilibrium one in the unperturbed system adds the condition that, with $a_6 = 0$, $f(\phi,t) = f_o(e^{-Yt}\phi)$ must be independent of t ; i.e.,

$$f_o(e^{-Yt}\phi) = f_o(\phi) \quad . \tag{9}$$

Since this is a condition on the functional form of the initial distribution and holds for all ϕ , (8) is thereby simplified to

$$f(\phi,t) = f_o\left(\phi - \int_o^t e^{Y\lambda} a_6(e^{-Y\lambda}\phi, t-\lambda)\ d\lambda\right) \quad . \tag{10}$$

This explicit expression for the time development of an equilibrium distribution under a perturbation $a_6(\phi, t)$ states that the distribution at phase ϕ at time t is given to first order by evaluating the equilibrium distribution $f_o(\phi)$ at the slightly displaced phase $\phi - \phi'$, where

$$\phi' = \phi'(\phi, t) = \int_o^t e^{Y\lambda} a_6(e^{-Y\lambda}\phi, t-\lambda)\, d\lambda \tag{11}$$

accumulates the perturbations suffered along the trajectory.

IV. INVERSE PHASE SPACE SPECTRA

Rather than proceeding to extract the velocity moments of the distribution function by integration of (10) to obtain the electromagnetic sources, the entire system description will be relegated to inverse phase space. This facilitates the extraction of velocity moments, and thence the electromagnetic fields, and averts the appearance of singularities corresponding to resonant particles.

Inverse phase space is spanned by the six-vector θ , composed of the wave vector $\underset{\sim}{k}$ and the inverse velocity vector $\underset{\sim}{\Lambda}$. The inverse phase space spectrum of the distribution is just its six-dimensional Fourier transform:

$$F(\theta,t) = \int f(\phi,t)\, e^{i\theta\cdot\phi} d^6\phi \quad . \tag{12}$$

The spectral description is algebraically simpler, replaces convolutions by products and drifts by phase factors, and yields the velocity moments of the distribution by merely evaluating the spectrum and its derivatives at the origin in inverse velocity space.[8] The distribution function description is best dispensed with entirely; the initial state of the system is to be specified directly in inverse phase space, by $F_o(\theta)$.

The evolution of the spectrum follows from that of the distribution function. Since

$$f(\phi,t) = f_o(\phi - \phi') = \int F_o(\theta_o)\, e^{-i\theta_o\cdot(\phi - \phi')} d^6\theta_o/(2\pi)^6 \quad , \tag{13}$$

expansion to first order in ϕ' and Fourier transformation yields for the perturbed spectrum

$$F(\theta,t) = F_o(\theta) + i\int F_o(\theta_o)\theta_o \cdot \theta'(\theta-\theta_o,t)\ d^6\theta_o/(2\pi)^6 \quad , \qquad (14)$$

where $\theta'(\theta,t)$ is the Fourier transform of $\phi'(\phi,t)$. From (11), this is given by

$$\theta'(\theta,t) = \int_o^t e^{\gamma\lambda}\int a_6(e^{-\gamma\lambda}\phi,t-\lambda)\ e^{i\theta\cdot\phi}\ d^6\phi\ d\lambda \quad , \qquad (15)$$

which indicates that the relevant perturbation quantity is the acceleration in inverse phase space, $A_6(\theta,t)$, the Fourier transform of $a_6(\phi,t)$. Since $d^6\phi = d^6(e^{-\gamma\lambda}\phi)$, (15) is expressible as

$$\theta'(\theta,t) = \int_o^t e^{\gamma\lambda}\ A_6(\theta\ e^{\gamma\lambda},\ t-\lambda)\ d\lambda \qquad (16)$$

and a similar change of variable, using $d^6\theta_o = d^6(\theta_o e^{\gamma\lambda})$ and the fact that the equilibrium condition, (9), translates into $F_o(\theta) = F_o(\theta e^{\gamma t})$, converts (14) into

$$F_1(\theta,t) = i\int_o^t\int F_o(\theta_\lambda-\theta_o)(\theta_\lambda-\theta_o)\cdot A_6(\theta_o,t-\lambda)d^6\theta_o d\lambda/(2\pi)^6 \quad . \qquad (17)$$

Here, $F_1(\theta,t)$ is the perturbation of the spectrum and

$$\theta_t = \theta\, e^{\gamma t} \tag{18}$$

may be viewed as the orbit in inverse phase space.

With the perturbation of the spectrum now expressed as a convolution in time, a final Laplace transformation is indicated, which will also reduce Maxwell's equations to algebraic ones. Denoting the Laplace transform operation by $G(s) = \underset{\sim s}{L}\, g(t)$, the convolution theorem yields for the transformed spectrum perturbation,

$$F_1(\theta,s) = i\underset{\sim s}{L} \int F_o(\theta_t-\theta_o)(\theta_t-\theta_o)\cdot A_6(\theta_o,s)\,d^6\theta_o/(2\pi)^6 \quad . \tag{19}$$

This relation specifies the operations to be carried out on the equilibrium spectrum $F_o(\theta)$ to obtain the system response to the perturbation.

Fig. 1 traces the methods of calculating the response of the system to the perturbing fields, in the phase-time, inverse phase-time, and inverse phase-frequency domains.

V. SELF-CONSISTENT FIELDS

The acceleration associated with a perturbation consists of any weak externally applied field, such as an impulse or a signal wave, together with the reaction provided by the internal electromagnetic fields: $A_6(\theta,s) = A_6^{ext}(\theta,s) + A_6^{em}(\theta,s)$. In particular, the plasma response to a weak impulse which imparts velocity $\underset{\sim}{v}_p$ uniformly to the particles is obtainable by setting

$$a_6(\phi,t) = \begin{bmatrix} 0 \\ v_p \end{bmatrix} \delta(t) \qquad A_6(\theta,s) = \begin{bmatrix} 0 \\ v_p \end{bmatrix} (2\pi)^6 \delta(\theta) \quad , \tag{20}$$

so that, from (19) and the equilibrium condition,

$$F_1(\theta,s) = i\underset{\sim}{L}_s \, F_o(\theta \, e^{Yt}) \; \theta \, e^{Yt} \begin{bmatrix} 0 \\ v_p \end{bmatrix} + F_1^{em}(\theta,s)$$

$$= iF_o(\theta) \; \theta\cdot(s - Y)^{-1} \begin{bmatrix} 0 \\ v_p \end{bmatrix} + F_1^{em}(\theta,s) \quad . \tag{21}$$

There remains to obtain the electromagnetic interaction.

Three steps are involved in expressing the internally generated perturbation $F_1^{em}(\theta,s)$ in terms of the total perturbation $F_1(\theta,s)$. By virtue of the present transform formulation, each of these has been reduced to algebraic manipulation. The first step is the extraction of the field sources from the perturbed spectrum; Maxwell's equations then yield the r.f. electromagnetic fields generated by these sources. Finally, the internally generated acceleration is expressed in terms of the perturbation through the Lorentz force. This closes the equations self-consistently.

If the system includes N particles of mass m and charge (-e) distributed in phase space as $f(\phi,t)$, then with the normalization $F_o(0) = 1$, the perturbation charge and current densities contributed by that species are given by their Fourier-Laplace transforms as

$$\rho_1(\underset{\sim}{k},s) = -e\, N\, F_1(\underset{\sim}{k},0,s) \;,\; \underset{\sim}{J}_1(\underset{\sim}{k},s) = ie\, N\, \partial\, F_1(\underset{\sim}{k},0,s)/\partial \underset{\sim}{\Lambda} \quad . \tag{22}$$

Note that particle conservation is expressed by

$$\frac{\underset{\sim}{k}}{s} \cdot \frac{\partial F_1(\underset{\sim}{k},0,s)}{\partial \underset{\sim}{\Lambda}} = F_1(\underset{\sim}{k},0,s) \quad . \tag{23}$$

Maxwell's equations combine into a wave equation, which transforms to

$$c^2\, \underset{\sim}{k} \times (\underset{\sim}{k} \times \underset{\sim}{\alpha}) = s^2\, \underset{\sim}{\alpha} - (e/m\epsilon_o)\; s\; \underset{\sim}{J}_1(\underset{\sim}{k},s) \quad , \tag{24}$$

where $\underset{\sim}{\alpha}(\underset{\sim}{k},s) = (-e/m)\, \underset{\sim}{E}(\underset{\sim}{k},s)$ is the transform of the acceleration suffered by a particle subjected to only the r.f. electric field, and $\underset{\sim}{J}_1(\underset{\sim}{k},s)$ is the total current density due to all the species. Continuing to consider only the contribution of the one species, (24) may be expressed in terms of the perturbed spectrum by

$$c^2(\underset{\sim}{k}\,\underset{\sim}{k} - k^2)\cdot\underset{\sim}{\alpha} = s^2\underset{\sim}{\alpha} - i\omega_o^2\; s\; \partial F_1(\underset{\sim}{k},0,s)/\partial\underset{\sim}{\Lambda} \quad , \tag{25}$$

where $\omega_o^2 = Ne^2/m\epsilon_o$ is the product of the average squared plasma frequency and the volume containing the N particles. Together with

(23), this equation incorporates Gauss' law,

$$\underset{\sim}{k}\cdot\underset{\sim}{\alpha} = i\omega_o^2 F_1(\underset{\sim}{k},0,s) \quad , \tag{26}$$

so that the solution of the wave equation is

$$\underset{\sim}{\alpha}(\underset{\sim}{k},s) = \frac{i\omega_o^2}{s^2 + k^2c^2}\left(c^2\underset{\sim}{k} + s\frac{\partial}{\partial\underset{\sim}{\Lambda}}\right)F_1(\underset{\sim}{k},0,s) \quad . \tag{27}$$

By Faraday's law, the transform of the acceleration due to the r.f. magnetic field interaction is $\underset{\sim}{v} \times (i\underset{\sim}{k} \times \underset{\sim}{\alpha})$, so that the total Lorentz acceleration is given in inverse phase space by

$$A_6(\underset{\sim}{k},\underset{\sim}{\Lambda},s) = \begin{bmatrix}0\\1\end{bmatrix}(2\pi)^3\left[\underset{\sim}{\alpha}\,\delta(\underset{\sim}{\Lambda}) + \left(\frac{\underset{\sim}{k}}{s}\underset{\sim}{\alpha} - \underset{\sim}{\alpha}\frac{\underset{\sim}{k}}{s}\right)\cdot\delta'(\underset{\sim}{\Lambda})\right] \quad . \tag{28}$$

It should be remarked that this formulation neglects relativistic corrections of order $(v/c)^2$ but retains all terms of order (v/c) .

Introducing $A_6(\underset{\sim}{k}_o,\underset{\sim}{\Lambda}_o,s)$ into (19) and integrating over $\underset{\sim}{\Lambda}_o$ leaves, after some manipulations,

$$F_1^{em}(\theta,s) = i\underset{\sim}{L}_s\int\left[F_t\,\underset{\sim}{\Lambda}_t + \underset{\sim}{G}_t \times (\underset{\sim}{k}_o/s)\right]\cdot\underset{\sim}{\alpha}(\underset{\sim}{k}_o,s)\,d^3k_o/(2\pi)^3 \quad , \tag{29}$$

where

$$\theta_t = (\underset{\sim}{k}_t,\underset{\sim}{\Lambda}_t) = \theta e^{Yt} = (\underset{\sim}{k},\underset{\sim}{\Lambda})e^{Yt} \;;\; F_t = F_o(\underset{\sim}{k}_t - \underset{\sim}{k}_o,\underset{\sim}{\Lambda}_t) \quad , \tag{30}$$

and

$$\underset{\sim}{G}_o(\underset{\sim}{k},\underset{\sim}{\Lambda}) = \underset{\sim}{\Lambda} \times \frac{\partial F_o(\underset{\sim}{k},\underset{\sim}{\Lambda})}{\partial \underset{\sim}{\Lambda}} \quad ; \quad \underset{\sim}{G}_t = \underset{\sim}{G}_o(\underset{\sim}{k}_t - \underset{\sim}{k}_o, \underset{\sim}{\Lambda}_t) \quad . \tag{31}$$

The first and second terms in the integral in (29) represent the r.f. electric and magnetic interactions, respectively, in the plasma. The equation gives the spectrum perturbation accompanying the electric field represented by $\alpha(\underset{\sim}{k},s)$, while (27) yields the field associated with the perturbation in the spectrum. In the absence of external excitation, the self-consistent fields are the solution of the homogeneous integral equation formed by combining (27) and (29), with $F_1^{em} = F_1$. The impulse response is given by the solution of the inhomogeneous integral equation obtained by combining (27), (29) and (21).

The primary objective of reducing the problem to the solution of a nonsingular integral equation has thus been attained. The kernel of the final integral equation is essentially the coefficient of $\alpha(\underset{\sim}{k}_o,s)$ in the integral in (29); it is formed from the function $F_o(\underset{\sim}{k},\underset{\sim}{\Lambda})$ representing the equilibrium spectrum by explicit operations specified in (30), (31), and (29). It is generally a well-behaved, nonsingular kernel. The integral equation ostensibly determines the plasma frequency, ω_o, as its eigenvalue. Since this is actually a specified parameter of the system, however, the determination of the eigenvalue effectively expresses the self-consistency condition, or the dispersion relation restricting s and $\underset{\sim}{k}$. The fact that ω_o^2 is a real, positive quantity imposes restrictions on the complex frequency s, which then stipulates the stability of the equilibrium state.

Summarizing the procedure for determining the stability, dispersion relation, or impulse response of a collection of particles, the system is first described by some inverse phase space spectrum $F_o(\theta)$, for each species. The force fields, including those externally applied and the internal ones which maintain the distribution, are described in terms of the accelerations produced and are assumed to compose a generalized harmonic oscillator, specified by the constant 6 × 6 matrix Y which validates (2). The restrictions on these initial specifications are the solenoidal field condition implied by Tr Y = 0 and the equilibrium condition expressed by $F_o(\theta e^{Yt}) = F_o(\theta)$.

The vector $\underset{\sim}{G}_o(\theta) = \underset{\sim}{\Lambda} \times \partial F_o/\partial\underset{\sim}{\Lambda}$ determines the magnetic interaction and essentially measures the anisotropy of the equilibrium velocity distribution. The orbit in inverse phase space, $(\underset{\sim}{k}_t, \underset{\sim}{\Lambda}_t) = (\underset{\sim}{k}, \underset{\sim}{\Lambda})e^{Yt}$, may be evaluated by standard matrix methods; the relation $e^{Yt} = \underset{\sim s}{L}^{-1}(s-Y)^{-1}$ probably yields this most easily. Then the perturbation of the spectrum is given in terms of the first-order electric field by (29), while the latter is given in terms of the former by (27). The combination forms an integral equation for $\underset{\sim}{\alpha}(\underset{\sim}{k},s)$, whose eigenvalues determine the dispersion relation among s, $\underset{\sim}{k}$, and the plasma frequency, ω_o . The response to applied signals is obtained from an inhomogeneous integral equation, formed as in (21) for the impulse response.

Fig. 2 depicts the processing of the given equilibrium spectrum and acceleration matrix required to arrive at the equations relating the spectrum and field perturbations.

To account for perturbations of several species, the individual fields of each may be superposed to form $\underset{\sim}{\alpha}$. Any small deviations from the harmonic oscillator equation, (2), may be treated as a further perturbation, as in (3).

VI. GAUSSIAN-MAXWELLIAN COLUMN

Before illustrating the method of setting up the integral equation with an application to a particular inhomogeneous magnetoplasma, it may be noted that for the homogeneous case, the integral equation is eliminated, reducing to an algebraic relation. This reduction is made possible physically by the fact that the particles then traverse only regions which respond to perturbations just as does the one in the immediate vicinity of the initial position, so that the global sensitivity is reduced to a local interaction. Formally, a homogeneous system is described by a spectrum of the form

$$F_o(\theta) = F_o(\underset{\sim}{k},\underset{\sim}{\Lambda}) = F(\underset{\sim}{\Lambda})\ (2\pi)^3\ \delta(\underset{\sim}{k}) \quad , \qquad (32)$$

whereupon (29) reduces to

$$F_1^{em}(\theta,s) = i\underset{\sim}{L}_s \left[F(\underset{\sim}{\Lambda}_t)\ \underset{\sim}{\Lambda}_t \cdot \underset{\sim}{\alpha}(\underset{\sim}{k}_t,s) + \underset{\sim}{G}(\underset{\sim}{\Lambda}_t)\cdot \frac{\underset{\sim}{k}_t}{s} \times \underset{\sim}{\alpha}(\underset{\sim}{k}_t,s) \right] \quad , \qquad (33)$$

where $\underset{\sim}{G}(\underset{\sim}{\Lambda}) = \underset{\sim}{\Lambda} \times \partial F/\partial \underset{\sim}{\Lambda}$. The self-consistent field equation is then obtained by combining this with (27), algebraically.

A fairly realistic example of a warm, inhomogeneous magnetoplasma is provided by one described by a Gaussian-Maxwellian spectrum:

$$F_o(\theta) = \exp\left(-\frac{1}{2}\ \theta\ R\ \theta\right) \quad . \qquad (34)$$

Here, R is a constant, symmetric 6 X 6 matrix, which partitions into four 3 × 3 submatrices as

$$R = \begin{bmatrix} R_{xx} & R_{xv} \\ R_{vx} & R_{vv} \end{bmatrix} , \tag{35}$$

incorporating the parameters that specify the mean spatial extent of the plasma, its thermal velocity, and its drift motion. As a matter of interest, for ease of visualization but superfluous to the calculation, this spectrum corresponds to the particle distribution function

$$f_o(\phi) = \exp\left(-\frac{1}{2}\,\phi R^{-1}\,\phi\right)/(2\pi)^3\,(\det R)^{1/2} , \tag{36}$$

which represents a collection of particles with a Gaussian spatial distribution and a Maxwellian velocity distribution. The particle drift is linear in position, $\underset{\sim}{u}(\underset{\sim}{x}) = R_{vx}\,R_{xx}^{-1}\,\underset{\sim}{x}$, and the temperature is uniform, specified by the mean squared thermal velocity, $T/m = v_\theta^2$:

$$v_\theta^2 = \frac{1}{3}\,\mathrm{Tr}(R_{vv} - R_{vx}\,R_{xx}^{-1}\,R_{xv}) . \tag{37}$$

These, and other velocity and spatial moments of the distribution, are readily obtainable by differentiation of the spectrum and evaluation at the origin of Λ- or k-space.

The equilibrium force field acting on the plasma is assumed to be expressible by the constant 6 × 6 matrix, Y , relating $d\phi/dt$ to ϕ , forming a generalized harmonic oscillator. This partitions into four

3×3 submatrices as

$$Y = \begin{bmatrix} 0 & I \\ Y_{vx} & Y_{vv} \end{bmatrix} . \qquad (38)$$

The submatrix Y_{vv} is to have zero trace to conserve the element of phase space. It represents an acceleration linear in velocity, as is the case for a magnetoplasma. Space charge fields are to be accounted for through Y_{vx} , which admits an acceleration linear in position, representing a particular class of ambipolar fields.

The condition for the unperturbed distribution to be an equilibrium state is here expressed by

$$F_o(\theta e^{Yt}) = \exp\left(-\frac{1}{2}\,\theta e^{Yt}\,Re^{Y't}\,\theta\right) = F_o(\theta) = \exp\left(-\frac{1}{2}\,\theta\,R\,\theta\right) \quad ,(39)$$

where the prime denotes the transpose of the matrix. By inspection of the quadratic form, it is readily seen that the equilibrium condition reduces to the requirement that YR be antisymmetric.

Specializing to a uniaxially symmetric magnetoplasma column, uniform in the axial direction but transversely Gaussian, reduces the 6×6 matrices to 4×4, eliminates the integration over the axial wave number, and requires N to be redefined as the axial linear particle density. A suitable combination of R and Y matrices that satisfy all the conditions is specified for electrons by the 2×2 submatrices

$$R_{xx} = a_e^2\, I \ , \ R_{vv} = v_e^2\, I \ , \ R_{xv} = R_{vx} = 0 \quad ;$$

$$Y_{vx} = -\omega_e^2\, I \ , \ Y_{vv} = \omega_{ce}\, X \ ; \ X = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} . \qquad (40)$$

The last of these effects the cross product operation with the axial magnetic field; ω_{ce} is the electron cyclotron frequency. There is thereby described a magnetoplasma column with Gaussian radial density profile of effective radius a_e and Maxwellian velocity distribution with thermal velocity v_e . Equilibrium obtains when $\omega_e = v_e/a_e$; this transit frequency corresponds to a time scale of the order of the time it would take a thermal electron to traverse the effective radius of the column.

Fig. 3 presents the essential characteristics of the Gaussian-Maxwellian magnetoplasma column here considered.

The configuration defined by (40) includes no particle drift but implies an ambipolar field providing a radially inward acceleration $(-\omega_e^2 r)$ which maintains the Gaussian profile. This field is to be provided by the interaction with the ion distribution. But this radially increasing ambipolar field which is to make the equilibrium configuration a harmonic oscillator strictly requires charges at infinity, or violates overall neutrality. This system is hence not strictly realizable on a permanent basis of macroscopic self-consistency. It is however possible to maintain the desired conditions, including charge neutrality, at least temporarily, by an appropriate selection of the ion distribution, as follows.

If the same number, N , of ions per unit length be distributed with a Gaussian profile of effective radius a_i slightly smaller than a_e , then the resulting ambipolar electric field would provide an inward acceleration to the electrons, of magnitude

$$\frac{\omega_o^2}{2\pi r}\left[\exp\left(-\frac{1}{2}\frac{r^2}{a_e^2}\right) - \exp\left(-\frac{1}{2}\frac{r^2}{a_i^2}\right)\right] . \qquad (41)$$

Near the axis of the column, where most of the electrons are concentrated, this does increase radially, at a rate which may be adjusted to the desired slope to yield the equilibrium condition.

$$[illegible]$$

The ion distribution defined by this effective radius a_i will make at least the inner core of the plasma column a harmonic oscillator in equilibrium, although the few electrons remote from the axis will undergo accelerations that differ slightly from those of a pure harmonic oscillator. These anharmonic accelerations may be treated as an additional perturbation on the equilibrium system and hence may be accounted for to first order in this formalism, with only minor modifications. The corrections are the smaller the more uniform the radial distribution and the lower the temperature.

The ambipolar field which here restrains the electrons tends to disrupt the ion distribution, which hence fails to conform to the postulated harmonic oscillator dynamics. The initial state here described is hence only a quasi-equilibrium state but the large ion-to-electron mass ratio ensures that the time development of the distribution can be quite slow compared to the time scale of the r.f. perturbations here considered. Since collisions were excluded at the outset, the validity of the formulation extends in time for only a mean collision time. It follows that the lack of permanent equilibrium is no severe restriction here since all that need be imposed is that the parameters allow the Gaussian distribution to persist essentially undistorted, in the absence of the perturbations, for one collision mean free time. The description is exact for a plasma which is either cold or homogeneous; it is a close approximation otherwise, provided the inhomogeneity and temperature are not extreme and the mass ratio is large.

With the understanding that the system is being observed only during that time interval in which it behaves as a harmonic oscillator in equilibrium, the analysis proceeds by specializing the general result to the spectrum

$$F_o(\theta) = \exp - \frac{1}{2} (k_\perp^2 a_e^2 + \Lambda^2 v_e^2) 2\pi \ \delta(k_\parallel) \quad , \tag{43}$$

and to the orbit in inverse phase space given axially by

$$(k_t, \Lambda_t)_\parallel = (k_\parallel, \Lambda_\parallel + k_\parallel t) \quad , \tag{44}$$

corresponding to free particle motion along the magnetic field, and transversely by

$$(k_t, \Lambda_t)_\perp = \underset{\sim}{L}_s^{-1} (k, \Lambda)_\perp \begin{bmatrix} s & -I \\ \omega_e^2 & s-\omega_{ce} X \end{bmatrix}^{-1} , \tag{45}$$

which involves the magnetic and ambipolar fields. Since the velocity distribution is here isotropic, $\underset{\sim}{G}_o(\theta) = 0$ and there is no first-order r.f. magnetic interaction. The spectrum perturbation is hence given in terms of the perturbing acceleration $\underset{\sim}{\alpha}(\underset{\sim}{k}, s)$ by

$$F_1^{em}(\theta, s) = \frac{i}{2\pi \ a_e^2} \underset{\sim}{L}_s \ e^{- \frac{1}{2} \Lambda_t^2 v_e^2} \ \underset{\sim}{\Lambda}_t \cdot \underset{\sim}{\gamma}(\underset{\sim}{k}_t, s) \quad , \tag{46}$$

where $\underset{\sim}{\gamma}(\underset{\sim}{k})$ is a Gaussian transform of $\underset{\sim}{\alpha}(\underset{\sim}{k})$:

$$\underset{\sim}{\gamma}(\underset{\sim}{k},s) = \int \left[\frac{a_e^2}{2\pi} \exp - \frac{1}{2} (\underset{\sim}{k}-\underset{\sim}{k}_o)^2 a_e^2\right] \underset{\sim}{\alpha}(\underset{\sim}{k}_o, k_{\parallel}, s) d^2k_o \quad . \tag{47}$$

The axial homogeneity gives the axial component $k_{\parallel}$ the role of a parameter. Note that in the fully homogeneous case, $a_e \to \infty$ and $\underset{\sim}{\gamma}(\underset{\sim}{k}) = \underset{\sim}{\alpha}(\underset{\sim}{k})$.

The integral equation for the self-consistent field $\underset{\sim}{\alpha}(\underset{\sim}{k},s)$ is now obtained by combining (46) with (27). Defining the plasma frequency ω_p in terms of the axial density by

$$\omega_p^2 = \omega_o^2/2\pi a_e^2 = Ne^2/2\pi a_e^2 m\epsilon_o \quad , \tag{48}$$

yields the result

$$\underset{\sim}{\alpha}(\underset{\sim}{k},s) + \frac{\omega_p^2}{s^2 + k^2c^2}\left(c^2\underset{\sim}{k} + s\frac{\partial}{\partial\underset{\sim}{\Lambda}}\right)\underset{\sim}{L}_s \left[c^{-\frac{1}{2}\Lambda_t^2 v_e^2}\, \underset{\sim}{\Lambda}_t \cdot \underset{\sim}{\gamma}(\underset{\sim}{k}_t, s)\right]_{\Lambda=0} = 0. \tag{49}$$

A more explicit form of this integral equation is derivable by use of the equilibrium condition, which here imposes

$$k_t^2 a_e^2 + \Lambda_t^2 v_e^2 = k^2 a_e^2 + \Lambda^2 v_e^2 + k_{\parallel}^2 v_e^2 t^2 + 2\Lambda_{\parallel} k_{\parallel} v_e^2 t \quad . \tag{50}$$

Substitution in (49) reduces the integral equation to

$$\underset{\sim}{\alpha}(\underset{\sim}{k},s) + \frac{\omega_p^2}{s^2 + k^2c^2} \underset{\sim}{L}_s \, e^{-\frac{1}{2} k_{\parallel}^2 v_e^2 t^2} \left(c^2 \underset{\sim}{k} - v_e^2 s t \underset{\sim}{k}_{\parallel} + s \frac{\partial}{\partial \underset{\sim}{\Lambda}} \right)_{\underset{\sim}{\Lambda}=0}$$

$$e^{-\frac{1}{2}(k_{\perp}^2 - k_t^2) a_e^2} \underset{\sim}{\Lambda}_t \cdot \underset{\sim}{\gamma}(\underset{\sim}{k}_t, s) = 0 \quad . \tag{51}$$

An alternate form involving an exponential rather than a Gaussian kernel follows from this by defining

$$\underset{\sim}{\beta}(\underset{\sim}{k},s) = \exp\left(-\frac{1}{2} k_{\perp}^2 a_e^2\right) \underset{\sim}{\alpha}(\underset{\sim}{k},s) \quad . \tag{52}$$

The unknown is now the convolution of the electric field with the density distribution and the integral equation becomes

$$\underset{\sim}{\beta}(\underset{\sim}{k},s) + \frac{\omega_p^2}{s^2 + k^2c^2} e^{-k_{\perp}^2 a_e^2} \underset{\sim}{L}_s \, e^{-\frac{1}{2} k_{\parallel}^2 v_e^2 t^2} \left(c^2 \underset{\sim}{k} - v_e^2 s t k_{\parallel} + s \frac{\partial}{\partial \underset{\sim}{\Lambda}} \right)_0 \underset{\sim}{\Lambda}_t \cdot$$

$$\int e^{a_e^2 \underset{\sim}{k}_t \cdot \underset{\sim}{k}_o} \underset{\sim}{\beta}(\underset{\sim}{k}_o, s) \frac{d^2(k_o a_e)}{2\pi} = 0 \quad . \tag{53}$$

Further insight into the properties of these integral equations may be gained by examining the explicit orbit in inverse phase space. Developing (45) yields for the transverse motion complementing (44),

$$k_{t\perp} = k_{\perp} Z(t) + \Lambda_{\perp} dZ/dt \ , \ \Lambda_{t\perp} = (-1/\omega_e^2) dk_{t\perp}/dt \ , \tag{54}$$

where $Z(t)$ is determined by a matrix partial fraction expansion of (45) as

$$\underset{\sim}{L}_s Z(t) = \frac{\omega_2}{\omega_1 + \omega_2}(s + \omega_1 X)^{-1} + \frac{\omega_1}{\omega_1 + \omega_2}(s - \omega_2 X)^{-1} \ , \tag{55}$$

or

$$Z(t) = \frac{\omega_2}{\omega_1 + \omega_2} \exp(-\omega_1 tX) + \frac{\omega_1}{\omega_1 + \omega_2} \exp(\omega_2 tX) \ . \tag{56}$$

This combines two rotations, in opposite senses, at rates whose difference is the cyclotron frequency and whose geometric mean is the transit frequency:

$$\omega_2 - \omega_1 = \omega_{ce} \ , \ \omega_1 \omega_2 = \omega_e^2 \ . \tag{57}$$

The trigonometric form of (56) is

$$Z(t) = \frac{\omega_1 \cos \omega_2 t + \omega_2 \cos \omega_1 t}{\omega_1 + \omega_2} + \frac{\omega_1 \sin \omega_2 t - \omega_2 \sin \omega_1 t}{\omega_1 + \omega_2} X \ , \tag{58}$$

which indicates that the kernel in (53) actually involves Bessel functions.[9] At $\underset{\sim}{\Lambda} = 0$, for $\underset{\sim}{k}\cdot\underset{\sim}{k}_o = kk_o \cos\phi$,

$$e^{a_e^2 \underset{\sim}{k}_t \cdot \underset{\sim}{k}_o} = \exp a_e^2 \underset{\sim}{k}\cdot Z(t)\cdot \underset{\sim}{k}_o$$

$$= \exp a_e^2 kk_o \left[\frac{\omega_1}{\omega_1 + \omega_2} \cos(\omega_2 t + \phi) + \frac{\omega_2}{\omega_1 + \omega_2} \cos(\omega_1 t - \phi) \right]$$

$$= \sum_{n,m} I_m\left(a_e^2 kk_o \frac{\omega_1}{\omega_1 + \omega_2}\right) I_n\left(a_e^2 kk_o \frac{\omega_2}{\omega_1 + \omega_2}\right) e^{i(m-n)\phi}\, e^{i(n\omega_1 + m\omega_2)t} . \tag{59}$$

Introducing the orbit into (53) reduces the integral equation to

$$\underset{\sim}{\beta}(\underset{\sim}{k},s) = \frac{\omega_p^2/\omega_e^2}{s^2 + k^2c^2}\, e^{-k_\perp^2 a_e^2}\, \underset{\sim}{L}_s\, e^{-\frac{1}{2} k_\parallel^2 v_e^2 t^2} \int H(\underset{\sim}{k},\underset{\sim}{k}_o,s,t)$$

$$\cdot\, e^{a_e^2 \underset{\sim}{k}\cdot Z\cdot \underset{\sim}{k}_o}\, \underset{\sim}{\beta}(\underset{\sim}{k}_o,s)\, \frac{d^2(k_o a_e)}{2\pi} , \tag{60}$$

where

$$H(\underset{\sim}{k},\underset{\sim}{k}_o,s,t) = (c^2\underset{\sim}{k} - v_e^2 st\underset{\sim}{k}_\parallel)\, \underset{\sim}{k}\, \frac{dZ}{dt} + s\left(\frac{d^2Z}{dt^2} + \frac{dZ}{dt}\, \underset{\sim}{k}_o a_e^2\, \underset{\sim}{k}\, \frac{dZ}{dt}\right) \tag{61}$$

is merely linear in $\underset{\sim}{k}_o$.

The above analysis illustrates the method of arriving at the integral equation governing an inhomogeneous magnetoplasma and the forms which that equation can take. The numerical analysis which remains to be carried out on the equation to extract the dispersion relation from the eigenvalues involves computations on only well-behaved kernels. While the final numerical processing of the equations to reveal the wave supporting and stability properties of the plasma here considered is beyond the scope of this expository work, a few general remarks on the character of the integral equations may be in order.

Several distinct but equivalent forms of the final integral equation have been presented. The kernel may be Gaussian, as in (51), or essentially exponential, as in (60), or a superposition of Bessel functions, as when the expansion (59) is substituted into (60). The choice permitted by the commutativity of the integral operator and the Laplace transformation in any of these forms provides further latitude in the selection of the equation to be analyzed numerically. For example, in (60), the Laplace transformation operates on the time function indicated, which involves $H(\underset{\sim}{k},\underset{\sim}{k}_o,s,t)$, whose Laplace transform is trivially deduced from (55), modified by what is purely an exponential function of t if (59) is introduced into (60). In the absence of the factor $\exp\left(-\frac{1}{2}k_{\parallel}^2 v_e^2 t^2\right)$, these exponentials would result in merely a shift in the frequency domain, with the Laplace transform of H evaluated at $\left[s - i(n\,\omega_1 + m\,\omega_2)\right]$. The effect of the exponential involving the axial wave vector $k_{\parallel}$ is to introduce a Gaussian factor in the transformation, leading to the familiar error function, or plasma dispersion function,[10] and to Landau damping.

The connection to the physical system is made through the factor (ω_p^2/ω_e^2) in (60), which plays the role of the eigenvalue of the integral equation but effectively relates the parameters s and $k_{\parallel}$, thereby expressing the dispersion relation. The stability of the equilibrium state is determined by those locations of s in the complex plane which permit the eigenvalue to be real and positive.

The two-dimensional character of the integral equations can be overcome in this case by taking advantage of the symmetry of the column to eliminate the azimuthal integration. By use of (59), this reduces the double summation to a single one, as well as leaving only a one-dimensional integration over the radial wave number k_o, suitable for digital computation.

Various approximations of both a mathematical and physical nature may be imposed on the equations. The common quasistatic approximation may be introduced by formally letting $c \to \infty$. In (60), this reduces $H(\underset{\sim}{k},\underset{\sim}{k}_o,s,t)/(s^2 + k^2c^2)$ to $(\underset{\sim}{k}\,\underset{\sim}{k}/k^2)dZ/dt$ and considerably simplifies the equation. Other simplifications are effected by selecting certain directions of wave propagation and polarization with respect to the magnetic field, particularly $k_{\parallel} = 0$. The combined case is treated more thoroughly in the Appendix.

For a mildly inhomogeneous and warm plasma, the transit frequency ω_e is low and the effective radius a_e is large. Suitable Taylor expansions of the kernels then permit the determination of at least the first-order deviations from the homogeneous case. It is already clear from (55-59) that to the very lowest order, the cyclotron frequency and its harmonics are replaced by the two frequencies ω_1, ω_2 and all their combination frequencies; ω_2 differs from ω_{ce} by ω_1, which is small for low transit frequencies. Beyond that, approximate results are obtainable by noting that the Gaussian factor in (47) becomes sharply peaked as $a_e \to \infty$. A Taylor expansion of $\underset{\sim}{\alpha}(\underset{\sim}{k}_o)$ about $\underset{\sim}{k}_o = \underset{\sim}{k}$ yields

$$\underset{\sim}{\gamma}(\underset{\sim}{k}) = \underset{\sim}{\alpha}(\underset{\sim}{k}) + \frac{1}{2a_e^2}\frac{\partial^2 \underset{\sim}{\alpha}}{\partial k^2} + \dots \quad . \tag{62}$$

When this is introduced into (51), there results a differential equation as an approximation to the integral equation.

Finally, it may be surmised from a qualitative examination of the integral equations with Gaussian kernels that whereas plane wave solutions of the form $\delta(\underset{\sim}{k} - \underset{\sim}{k}_s)$ are appropriate in the limiting infinite, homogeneous case, the solutions for $\underset{\sim}{\alpha}(\underset{\sim}{k})$ will in the general case involve a spectrum of $\underset{\sim}{k}$ vectors, sharply peaked about some signal wave vector $\underset{\sim}{k}_s$. The dispersion relation then relates the signal frequency to this central vector of the wave packet, which is however comprised of a broadened spectrum of $\underset{\sim}{k}$ vectors.

VII. CONCLUSIONS

The self-consistent field problem for a class of anisotropic, inhomogeneous particle distributions has been reduced to that of solving an integral equation with a well-behaved kernel. Tractability has been achieved mainly by considering a class of systems for which the equilibrium particle orbits can be simply described analytically. In particular, systems whose dynamics correspond to small deviations from generalized harmonic oscillation have been analyzed. The particle propagator then has the form $\exp Yt$, making it relatively simple to evaluate and accumulate the perturbations along the orbit.

The approach can evidently be generalized to other systems with known and simply expressed unperturbed particle orbits. A rather direct generalization might be to certain time-varying systems, in which the propagator has the form $\exp \int_0^t Y(\tau)\, d\tau$, provided the acceleration matrix $Y(t)$ commutes with its integral. A more sweeping generalization might be made to orbit equations quadratic in the phase, rather than merely linear. The orbit then satisfies a Riccati equation and the systems so described admit ambipolar fields more general than those considered herein.

Simple particle orbits permit simple recipes for the evolution of an arbitrary initial particle distribution function in response to arbitrary perturbations. The relevant physical quantities are, however, various moments of the distribution, or other expectation values. Their extraction by integration over velocity space is not only cumbersome but complicated by classes of particles whose orbits are in resonance with the propagation of disturbances. These difficulties are avoided by describing the system at the outset by its spectrum in inverse phase space. This spectrum is the characteristic function of the distribution, or the expectation value of $\exp i\theta\cdot\phi$, and exhibits all the moments by its behavior in the immediate vicinity of the origin of inverse phase space.

The perturbation of the spectrum is expressed in terms of the Fourier-Laplace transform of the perturbation, which is algebraically related to the sources of the electromagnetic fields, which are in turn given directly by the perturbed spectrum. The system of equations is thus closed self-consistently through an integral equation whose kernel is essentially the unperturbed spectrum evaluated along the orbit in inverse phase space. The equation admits arbitrary combinations of equilibrium spectra and the associated harmonic oscillator force fields and involves well-behaved kernels, well-suited for final numerical analysis.

The case of a Gaussian-Maxwellian plasma has been treated in greater detail, by way of illustration. The plasma configurations conforming to the $\exp(-\frac{1}{2}\theta R \theta)$ spectrum have not been exhausted herein. One which requires no ambipolar electric field to maintain its density profile is a column described transversely by

$$R_4 = a^2 \begin{bmatrix} I & -\omega X \\ \omega X & \omega\omega_c \end{bmatrix} . \qquad (63)$$

This has a Gaussian radial profile with effective radius a, a Maxwellian velocity distribution of temperature $T = m\, a^2 \omega(\omega_c - \omega)$, and an azimuthal drift corresponding to rotation at the rate ω about the axis of the column.[11] Except for diamagnetic effects of the azimuthal drift current, this column is in equilibrium in an axial magnetic field specified by the cyclotron frequency ω_c. The foregoing analysis applies to this configuration, with R replaced by (63), $Y_{vx} = 0$, $Y_{vv} = \omega_c X$. It may also be noted here that displacements through $\underset{\sim}{x}_o$ and drifts at velocity $\underset{\sim}{v}_o$ can be simply incorporated into the inverse phase space spectra, through a phase factor $\exp i\theta \cdot \phi_o$.

The illustrative example presented here affords an opportunity to formulate integral equations for a plasma in which temperature, anisotropy, inhomogeneity, particle transport, and ambipolar fields are all to be accounted for. There appears to be no insurmountable obstacle, other than increased complexity, to generalizing the formalism to include collisions and relativistic effects.

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APPENDIX

TRANSVERSE OSCILLATIONS OF MAGNETOPLASMA COLUMN

Under the quasistatic approximation, the special case of a self-consistent, azimuthally symmetric transverse field pattern, which may represent the cutoff condition for axial propagation along the Gaussian-Maxwellian magnetoplasma column, is sufficiently simple to analyze quite thoroughly. With reference to (52,60,61), the conditions are expressed by $c \to \infty$, $k_{\parallel} = 0$, $\underset{\sim}{\alpha}(\underset{\sim}{k},s) = \underset{\sim}{n}h(u,s)$, where $\underset{\sim}{n}$ is a unit vector along $\underset{\sim}{k}$ and $u = ka_e$. It then suffices to set $\Lambda = 0$, so that the orbit is given by $\underset{\sim}{k}_t = \underset{\sim}{k}Z(t)$ and the integral equation reduces to

$$h(u,s) = \frac{\omega_p^2}{\omega_e^2} \frac{e^{-\frac{1}{2}u^2}}{u} \underset{\sim}{L}_s \frac{d}{dt} \int \frac{e^{-\frac{1}{2}u_o^2}}{u_o} e^{uu_o \underset{\sim}{n}Z\underset{\sim}{n}_o} h(u_o,s)\frac{d^2u_o}{2\pi} \quad . \tag{A-1}$$

Choosing polar coordinates with polar axis along $\underset{\sim}{n}$ and azimuth defined by $\underset{\sim}{n}_o = \exp(\phi X)\underset{\sim}{n}$, the azimuthal integration becomes, using (56),

$$\int_o^{2\pi} e^{uu_o \underset{\sim}{n}Z\underset{\sim}{n}_o} \frac{d\phi}{2\pi} = \int_o^{2\pi} \exp\left(uu_o\left[\frac{\omega_1}{\omega_3}\cos(\phi - \omega_1 t) + \frac{\omega_2}{\omega_3}\cos(\phi + \omega_2 t)\right]\right)\frac{d\phi}{2\pi}$$

$$= I_o(u\Omega(t)u_o) \quad , \tag{A-2}$$

where

$$\Omega^2(t) = \left(\omega_1^2 + 2\omega_1\omega_2 \cos \omega_3 t + \omega_2^2\right)\Big/\omega_3^2 \quad , \qquad \omega_3 = \omega_1 + \omega_2 \quad . \tag{A-3}$$

Consequently, the integral equation collapses to

$$h(u,s) = \lambda (e^{-\frac{1}{2}u^2}/u) \underset{\sim s}{L}(d/dt)\int_0^\infty e^{-\frac{1}{2}u_o^2} I_o(u\Omega(t)u_o)\, h(u_o,s)du_o \ , \tag{A-4}$$

where $\lambda = \omega_p^2/\omega_e^2$ is the eigenvalue.

By use of the multiplication theorem for Bessel functions,[12] it is possible to reduce the kernel of this equation to degenerate form. Since

$$I_o(u\Omega u_o) = \sum_{m=0}^{\infty} \frac{(\Omega^2 u^2 - 1)^m}{2^m\ m!}\ u_o^m I_m(u_o) \ , \tag{A-5}$$

the equation is equivalent to

$$h(u,s) = \lambda \frac{e^{-\frac{1}{2}u^2}}{u} \underset{\sim s}{L} \frac{d}{dt} \sum_{m=0}^{\infty} \frac{(\Omega^2 u^2 - 1)^m}{2^m\ m!} H_m(s) \ , \tag{A-6}$$

where

$$H_m(s) = \int_0^\infty e^{-\frac{1}{2}u^2} I_m(u)\ u^m h(u,s)du \ . \tag{A-7}$$

The degeneracy now permits the reduction of the integral equation to an equivalent infinite set of simultaneous equations of the form

$$H_n(s) = \lambda \sum_{m=1}^{\infty} G_{nm}(s)\ H_m(s) \ , \tag{A-8}$$

with

$$G_{nm}(s) = \frac{1}{2^m\, m!} \int_0^\infty e^{-u^2} I_n(u) u^{n-1} \underset{\sim s}{L} \frac{d}{dt} (\Omega^2 u^2 - 1)^m du \quad . \qquad \text{(A-9)}$$

These matrix elements may be evaluated explicitly, by virtue of the relations

$$\underset{\sim s}{L} \frac{d}{dt} \Omega^{2k} = \sum_{p=1}^{k} \binom{k}{p} (-1)^p (2p)! \prod_{q=1}^{p} \left(\frac{\omega_e^2}{s^2 + q^2 \omega_3^2} \right) , \qquad \text{(A-10)}$$

and, in terms of Laguerre polynomials,

$$\int_0^\infty e^{-u^2} I_n(u) u^{n+2k-1} du = \frac{(k-1)!}{2^{n+1}} e^{\frac{1}{4}} L_{k-1}^n(-\tfrac{1}{4}) \quad . \qquad \text{(A-11)}$$

Thus,

$$G_{nm}(s) = \frac{e^{1/4}}{2^{n+m}} \sum_{k=1}^{m} \frac{(-1)^{m-k}}{(m-k)!} \frac{L_{k-1}^n(-\frac{1}{4})}{2k} \sum_{p=1}^{k} \binom{k}{p} (-1)^p (2p)! \prod_{q=1}^{p} \left(\frac{\omega_e^2}{s^2 + q^2 \omega_3^2} \right),$$

(A-12)

and the resonance condition is

$$\det\,(I - \lambda G) = 0 \quad , \qquad \text{(A-13)}$$

which relates the complex frequency s to the plasma frequency ω_p in $\lambda = \omega_p^2/\omega_e^2$. Clearly, the harmonics of $\omega_3 = (\omega_{ce}^2 + 4\omega_e^2)^{1/2}$ replace the cyclotron harmonics as the significant frequencies.

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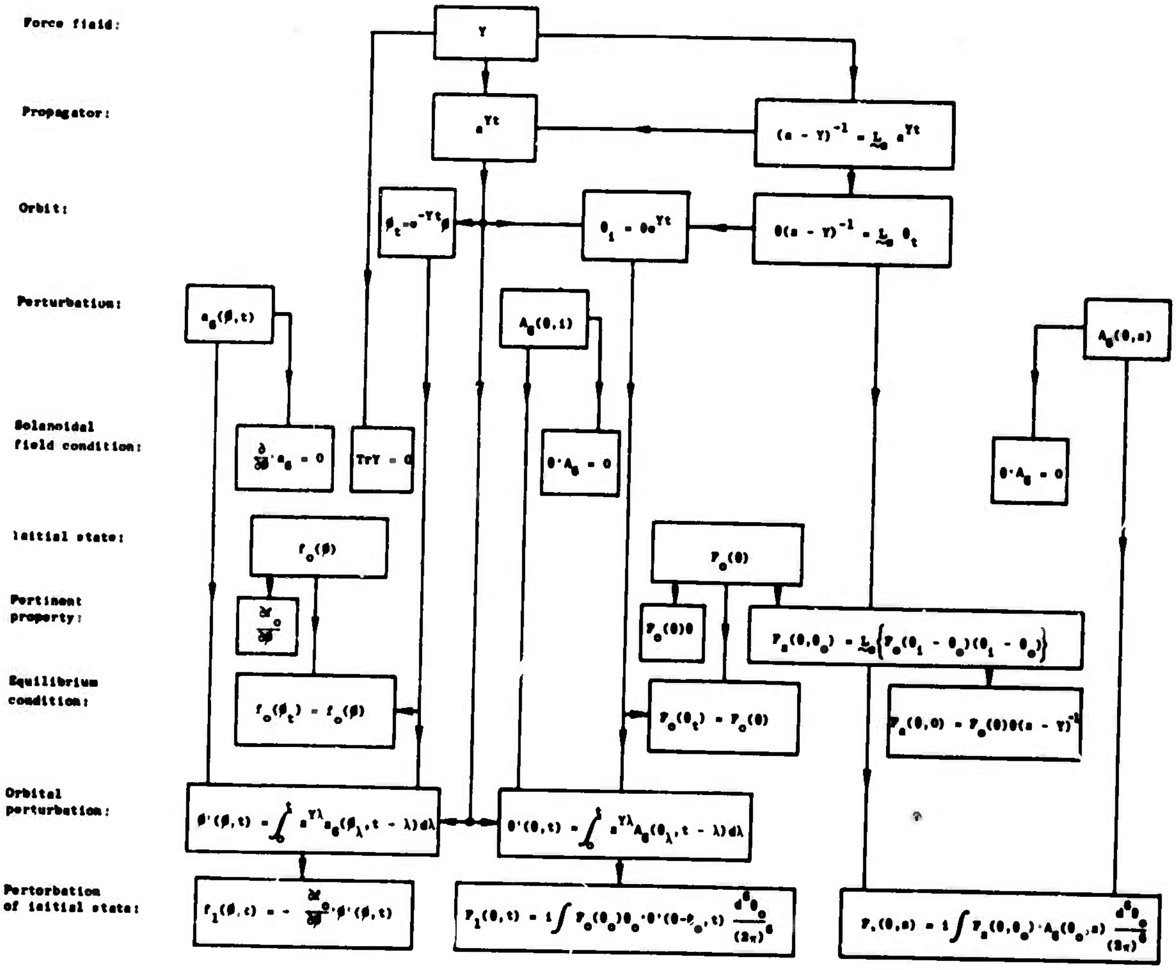


Fig. 1 - Comparison of calculation of response to perturbations in (ϕ,t) , (θ,t) , and (θ,s) spaces.

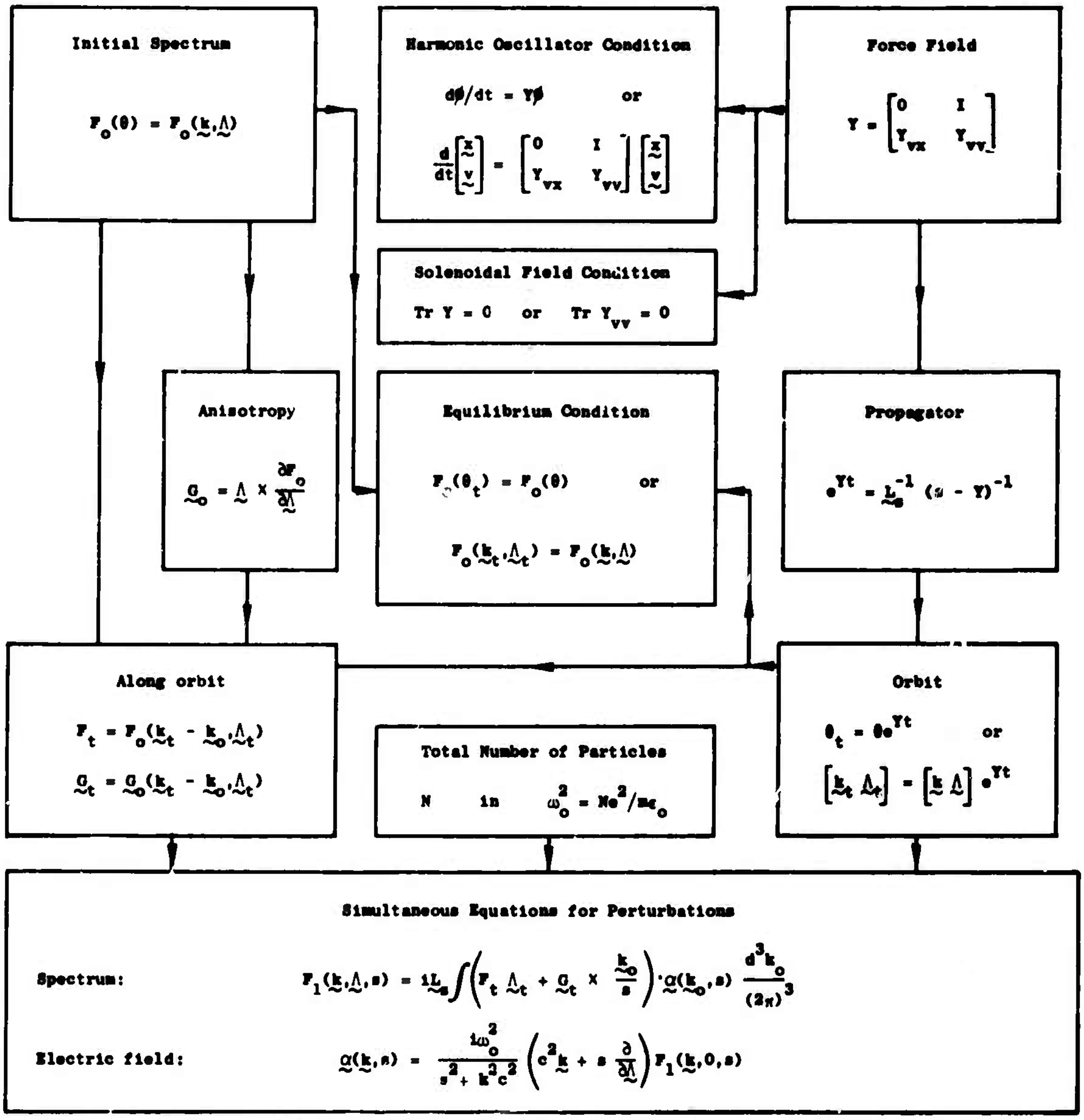


Fig. 2 - Evolution of self-consistent equations for spectrum and field perturbations.

Electron distribution and spectrum

$$f_o(\underset{\sim}{x},\underset{\sim}{v}) = \left[2\pi a_e^2\left(2\pi v_e^2\right)^{3/2}\right]^{-1} \exp - \frac{1}{2}\frac{m}{T}\left(v^2 + \omega_e^2 r^2\right)$$

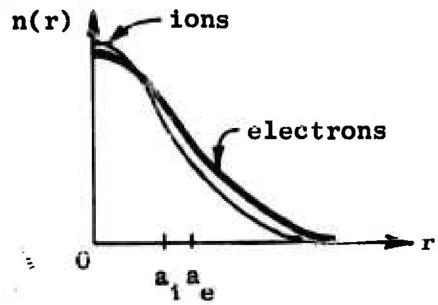
$$F_o(\underset{\sim}{k},\underset{\sim}{\Lambda}) = \exp - \frac{1}{2}\left(k^2 a_e^2 + \Lambda^2 v_e^2\right)$$

$$\omega_e = v_e/a_e \qquad T/m = v_e^2$$

Electron and ion density profiles

$$n_e(r) = \left(N/2\pi a_e^2\right) \exp\left(-\frac{1}{2}\frac{r^2}{a_e^2}\right)$$

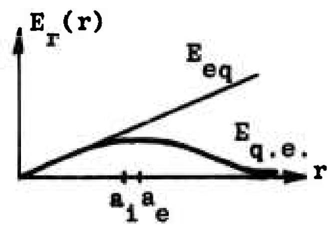
$$n_i(r) = \left(N/2\pi a_i^2\right) \exp\left(-\frac{1}{2}\frac{r^2}{a_i^2}\right)$$



Equilibrium and quasi-equilibrium electric fields

$$E_{eq.}(r) = \frac{Ne}{2\pi\epsilon_o}\left(\frac{1}{a_i^2} - \frac{1}{a_e^2}\right)\frac{r}{2}$$

$$E_{q.e.}(r) = \frac{Ne}{2\pi\epsilon_o}\left[\exp\left(-\frac{1}{2}\frac{r^2}{a_e^2}\right) - \exp\left(-\frac{1}{2}\frac{r^2}{a_i^2}\right)\right]/r$$



Quasi-equilibrium condition

$$\frac{\omega_o^2}{4\pi}\left(\frac{1}{a_i^2} - \frac{1}{a_e^2}\right) = \frac{v_e^2}{a_e^2} = \omega_e^2 = \frac{\omega_p^2}{2}\left(\frac{a_e^2}{a_i^2} - 1\right) \quad \text{or} \quad \left(\frac{a_e^2}{a_i^2} - 1\right) = 2\,\frac{\lambda_D^2}{a_e^2}$$

Fig. 3 - The Gaussian-Maxwellian magnetoplasma column.

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13. ABSTRACT

The problem of wave dispersion and stability for a class of hot, inhomogeneous, collisionless magnetoplasmas is reduced to the solution of an integral equation with well-behaved kernel. Admissible configurations include those for which the externally applied and internal ambipolar fields form a generalized harmonic oscillator. The full set of Maxwell's equations is used to arrive at self-consistent perturbation fields in terms of the equilibrium particle distributions. An illustrative example treats a magnetoplasma column with Gaussian radial profile and Maxwellian velocity distribution in a state of quasi-equilibrium.

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